Notes on Seiberg-Witten theory

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These are notes prepared for the graduate course on Supersymmetry and Supergravity offered in Fall 2015 by Professor Peter van Nieuwenhuizen. We study the Seiberg-Witten solution of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory in 3+1 dimensions with gauge group $SU(2)$ and no additional matter.

1 Introduction

Determining the infrared behaviour of quantum Yang-Mills theory is an outstanding problem in theoretical high energy physics. A real-world application is QCD. It is conjectured that the low-energy theory has a mass gap (''in the infrared'') and that particles with non-abelian charges are confined, something which is observed experimentally in QCD. The theory turns out to be strongly coupled in the infrared which makes a direct verification of these claims intractable.

The situation in supersymmetric Yang-Mills theories is much better since supersymmetry severely constrains quantum corrections in these theories. This is usually phrased in terms of non-renormalization theorems. For example, the superpotential in $\mathcal{N} = 1$ theories does not renormalise in perturbation theory. Another example is that in $\mathcal{N} = 2$ theories the β-function does not receive perturbative quantum corrections beyond one loop. Non-perturbative effects are under much less control, though they are also quite constrained by supersymmetry.

In 1994, Seiberg and Witten [SW1] attacked the problem of determining the infrared physics of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory with $SU(2)$ gauge group and no hypermultiplets. In a companion paper [SW2], they extended their analysis to the case with hypermultiplets. The dynamics of an $\mathcal{N} = 2$ SYM theory can be packaged into a single holomorphic function of the $\mathcal{N} = 2$ chiral superfields called *the prepotential*. Seiberg and Witten were able to completely determine the low-energy prepotential for the theory using indirect arguments that were made possible by $\mathcal{N} = 2$ supersymmetry where the anomaly in R-symmetry belongs to the same anomaly multiplet as the β-function (see Appendix A). As a byproduct of their solution, it was also possible to completely determine the BPS spectrum (the physical states for which the mass equals the modulus of the central charge). And most interestingly, confinement due to condensation of magnetic monopoles occurred in a natural way. It was possible, for the first time, to completely solve for the strongly coupled dynamics of a gauge theory and observe phenomena like confinement!

There exist many reviews in the literature where various aspects of the Seiberg-Witten solution have been worked out in detail and extended to more general gauge groups and theories with hypermultiplets. A few of these are [AH, Bi, Le, Ta, Ga]. In these notes, we shall elaborate on the Seiberg-Witten solution for gauge group $SU(2)$ without additional matter, with emphasis on the magic role played by $\mathcal{N} = 2$ supersymmetry.

In Section 2 we will review basic aspects of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory in $\mathcal{N} = 2$ superspace. In Section 3 we introduce the quantum moduli space $\mathcal{M}_q$ of $\mathcal{N} = 2$ SYM theory with gauge group $SU(2)$. This moduli space turns out to be the complex plane, parametrized by $u \in \mathbb{C}$. We study quantum corrections and see how these are constrained by $\mathcal{N} = 2$ supersymmetry. The theory is asymptotically free and there is a dynamically generated scale Λ (similar to Λ_{QCD}). We study the low-energy effective theory at different points of the moduli space and see that it is strongly coupled for $|u|$ smaller than Λ^2. Section 4 deals with non-perturbative quantum corrections and and their relevance for the problem at hand.

In Section 5, we develop the $\mathcal{N} = 2$ version of electric-magnetic duality and demonstrate how it can be used to study the low-energy effective theory in the strongly-coupled region. In Sections 6 and 7, we first obtain a formula for the quantum corrections to the central charge in terms of the electric and magnetic charges. Using this formula and electric-magnetic duality we argue, following Seiberg and Witten [SW1], that monopoles and dyons becomes massless at certain strongly coupled points in the moduli space. This allows us to determine the prepotential of the low energy-effective theory at all points of the quantum moduli space, hence solving the model.

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2 Super Yang-Mills theory in $\mathcal{N}=2$ superspace

2.1 $\mathcal{N}=2$ superspace and the field strength W

We choose coordinates $\{x^\mu, \theta^{aA}, \bar{\theta}^{\dot{A}}_a\}$ with $a=1,2$ for $\mathcal{N}=2$ superspace. Under complex conjugation, we define

$$(\theta^a_A)^* = \bar{\theta}_{a\dot{A}} \ , \quad (\theta^{aA})^* = -\bar{\theta}^{\dot{A}}_a \ . \tag{1}$$

We introduce susy-covariant derivatives D_{aA} and $\overline{D}^a_{\dot{A}}$:

$$\begin{aligned} D_{aA} &= \frac{\partial}{\partial\theta^{aA}} - i(\sigma^\mu)_{A\dot{B}}\,\bar{\theta}^{\dot{B}}_a\,\partial_\mu \ , \\ \overline{D}^a_{\dot{A}} &= \frac{\partial}{\partial\bar{\theta}^{\dot{A}}_a} - i(\sigma^\mu)_{B\dot{A}}\,\theta^{aB}\,\partial_\mu \ , \end{aligned} \tag{2}$$

and the gauge-covariant susy-covariant derivatives

$$\nabla_{aA} = D_{aA} + A_{aA} \ , \quad \overline{\nabla}^a_{\dot{A}} = \overline{D}^a_{\dot{A}} + A^a_{\dot{A}} \ , \quad \nabla_{A\dot{B}} = D_{A\dot{B}} + A_{A\dot{B}} \ , \tag{3}$$

with $D_{A\dot{B}} = (\sigma^\mu)_{A\dot{B}}\partial_\mu$. It follows from (1) that $(\nabla_{aA})^\dagger = -\overline{\nabla}^a_{\dot{A}}$ and $(\nabla_{A\dot{B}})^\dagger = -\nabla_{\dot{A}B} = -\nabla_{B\dot{A}}$.

To obtain the component fields of $\mathcal{N}=2$ SYM theory, we introduce an *a priori* unconstrained $\mathcal{N}=2$ superfield W which is Lie-algebra valued: $W=W^iT_i$ with T_i antihermitian. We then impose the following constraints on the gauge-covariant derivatives:

$$\{\nabla_{aA}\,,\nabla_{bB}\}=\frac{1}{2}\epsilon_{AB}\,\epsilon_{ab}\,\overline{W}\ ,\quad \{\overline{\nabla}^a_{\dot{A}}\,,\overline{\nabla}^b_{\dot{B}}\}=\frac{1}{2}\epsilon_{\dot{A}\dot{B}}\,\epsilon^{ab}\,W\ ,$$
$$\overline{W}=-W^\dagger\ ,\quad \{\nabla_{aA}\,,\overline{\nabla}^b_{\dot{B}}\}=-2i\delta^b_a\,\nabla_{A\dot{B}}\ . \tag{4}$$

The second relation follows from the first upon using that $\epsilon_{AB}=-\epsilon_{\dot{A}\dot{B}}$ and $\epsilon_{ab}=\epsilon^{ab}$. One may check the sign on the RHS in the last relation by substituting (2). For any superfield X we define $\overline{X}=-X^\dagger$. If X is Lie-algebra valued, $X=X^iT_i$, it follows that $\overline{X}=(X^i)^\dagger T_i$.

W is the field strength corresponding to the gauge-covariant derivatives $\overline{\nabla}^a_{\dot{A}}$. The following constraints on W follow from imposing the Bianchi identities on the gauge-covariant derivatives:

$$\boxed{\nabla^A_a\nabla_{bA}\,W=\overline{\nabla}^{\dot{A}}_b\overline{\nabla}_{a\dot{A}}\,\overline{W}\ ,\quad \overline{\nabla}_{a\dot{A}}W=0\ ,\quad \nabla_{aA}\overline{W}=0\ .} \tag{5}$$

The first constraint is a reality constraint and the last two are chirality constraints on W and $\overline{W}$. Thus, W is a $\mathcal{N}=2$ chiral superfield obeying a reality constraint. We shall see that these constraints are just sufficient to obtain the component fields of $\mathcal{N}=2$ SYM theory.

2.2 $\mathcal{N}=1$ field content in W

What is the field content of W after the above constraints are imposed? We first notice that taking $a=b=1$ gives us a $\mathcal{N}=1$ subalgebra with the $\mathcal{N}=1$ supercoordinates being $\theta^A\equiv\theta^{1A}$ and $\theta^{\dot{A}}\equiv\bar{\theta}^{\dot{A}}_1$. Thus, we can truncate $\mathcal{N}=2$ superspace to $\mathcal{N}=1$ superspace by setting $\theta^{2A}=0$, $\bar{\theta}^{\dot{A}}_2=0$, which we denote by a slash $|$. We define the following $\mathcal{N}=1$ superfields:

$$W_|\equiv\phi\ ,\qquad (\nabla_{2A}\,W)_|\equiv W_A\ ,\qquad (\nabla^A_2\nabla_{2A}W)_|\equiv G\ . \tag{6}$$

There are no other independent $\mathcal{N}=1$ superfields in W since W is $\mathcal{N}=2$ chiral. It is easy to see that ϕ is $\mathcal{N}=1$ chiral, $\overline{\nabla}_{\dot{A}}\phi=\overline{\nabla}^1_{\dot{A}}\phi=0$. Using the constraints on the gauge-covariant derivatives in (4), one can check that W_A is also $\mathcal{N}=1$ chiral. Next, using the reality constraint $\nabla^A_a\nabla_{bA}\,W=-\overline{\nabla}^{\dot{A}}_b\overline{\nabla}_{a\dot{A}}\,\overline{W}$ with $a=1$ and $b=2$, we obtain

$$\nabla^AW_A=\overline{\nabla}^{\dot{A}}\overline{W}_{\dot{A}}\ ,\quad \text{since}\quad \overline{\nabla}^{\dot{A}}_2=\overline{\nabla}^{1\dot{A}}=\overline{\nabla}^{\dot{A}}\,,\overline{\nabla}_{\dot{A}}=-\overline{\nabla}^2_{\dot{A}}\ ,$$
$$\text{and}\quad \overline{W}_{\dot{A}}=-(W_A)^\dagger=[W^\dagger\,,\overline{\nabla}^2_{\dot{A}}]_|=-[\overline{\nabla}^2_{\dot{A}}\,,W^\dagger]_|=\overline{\nabla}^2_{\dot{A}}\,\overline{W}_|=-\overline{\nabla}_{1\dot{A}}\,\overline{W}_|\ . \tag{7}$$

(Here, $\overline{\nabla}_{\dot{A}}^{a=2}$ is written as $\overline{\nabla}_{\dot{A}}^{2}$, and it is *not* the square of $\overline{\nabla}_{\dot{A}}$. Such squares will always be shown with the indices explicitly contracted, e.g. $\overline{\nabla}_a^{\dot{A}}\overline{\nabla}_{b\dot{A}}$.) Taking $a = b = 2$ in the reality constraint, we get

$$G = \overline{\nabla}^{\dot{A}}\overline{\nabla}_{\dot{A}}\,\bar{\phi}\ . \tag{8}$$

This shows that ϕ and W_A are the only independent $\mathcal{N}=1$ superfields in the constrained $\mathcal{N}=2$ superfield W and that G is **not** an independent $\mathcal{N}=1$ superfield. To summarise, the covariant constraints in (4) ensure that one obtains precisely the field content of $\mathcal{N}=2$ SYM with no extra degrees of freedom.

2.3 $\mathcal{N}=2$ SYM action in $\mathcal{N}=2$ superspace

The x-space action of pure Yang-Mills theory with a CP violating θ-angle (not to be confused with the supercoordinate θ^{aA}) is as follows:

$$\mathcal{S}_{\text{YM}} = \int \mathrm{d}^4x\ \mathrm{tr}\left[\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} - \frac{\theta}{16\pi^2}F_{\mu\nu}{}^{\star}F^{\mu\nu}\right]\ , \tag{9}$$

where ${}^{\star}F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ is the dual field strength. The two coupling constants g and θ can be combined into one complex coupling constant τ and the field strength and its dual can be combined into one complex "self-dual" field strength $\mathcal{F}_{\mu\nu}$[1]:

$$\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}\ ,\qquad \mathcal{F}_{\mu\nu} \equiv F_{\mu\nu} - i{}^{\star}F_{\mu\nu}\ . \tag{10}$$

and the action can be rewritten in terms of τ as follows:

$$\mathcal{S}_{\text{YM}} = \frac{1}{16\pi}\mathrm{Im}\int \mathrm{d}^4x\ \mathrm{tr}\left(\tau\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right)\ . \tag{11}$$

The $\mathcal{N}=2$ version of the above has a very similar form:

$$\mathcal{S}_{\text{SYM}} = -\frac{1}{16\pi}\mathrm{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta^1\mathrm{d}^2\theta^2\ \mathrm{tr}\left(\tau W^2\right)\ , \tag{12}$$

where W is the $\mathcal{N}=2$ chiral superfield introduced earlier. The super-integration is only over the chiral part of $\mathcal{N}=2$ superspace since W is chiral. To prove (12), we use the definition of $\mathcal{N}=1$ superfields given in (6), replace $\mathrm{d}^2\theta^2$ by $-\frac{1}{4}\nabla_2^A\nabla_{2A}$ inside the trace and carry out the D-algebra to obtain the following action in $\mathcal{N}=1$ superspace:

$$\mathcal{S}_{\text{SYM}} = \frac{1}{32\pi}\mathrm{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\ \mathrm{tr}\left[\tau\left(W^AW_A - 4\int \mathrm{d}^2\bar{\theta}\ \bar{\phi}\phi\right)\right]\ ,\quad \bar{\phi} = -\phi^{\dagger}\ . \tag{13}$$

[1] In Minkowski space, $\star^2 = -1$, which implies that the eigenspaces correspond to eigenvalues $\pm i$. Hence (anti) self-duality is to be interpreted as ${}^{\star}\mathcal{F} = \pm i\mathcal{F}$.

The above action is indeed the $\mathcal{N}=1$ SYM action coupled to an $\mathcal{N}=1$ WZ action (with the chiral superfield in the adjoint representation) and it has a second supersymmetry which transforms ϕ into W_A, $\bar{\phi}$ into $\overline{W}_{\dot{A}}$ and W_A into ϕ and $\bar{\phi}$. To see the coupling more clearly, we observe that the above action is written in what is known as the **vector representation**. There is another representation, the **chiral representation**, in which the coupling of WZ to SYM is more manifest. The transformation between the two representations is a change of basis in the space of $\mathcal{N}=1$ superfields. We first note that the $\mathcal{N}=1$ gauge-covariant derivatives have the following form in the vector representation after solving the constraints:

$$\nabla_A = e^{-\Omega} D_A\, e^{\Omega} \ , \quad \overline{\nabla}_{\dot{A}} = e^{-\bar{\Omega}} \overline{D}_{\dot{A}}\, e^{\bar{\Omega}} \ , \tag{14}$$

where Ω is a Lie-algebra valued superfield with $\Omega = \Omega^i T_i$ and $\overline{\Omega} = (\Omega^i)^\dagger\, T_i$. In other words, $\Omega^\dagger = -\overline{\Omega}$ since the T_i are taken to be antihermitian. The $\mathcal{N}=1$ *real* superfield V (with $V = V^\dagger = -\overline{V}$) which contains the component fields of $\mathcal{N}=1$ SYM theory is defined in terms of Ω as

$$e^V \equiv e^{\Omega} e^{-\bar{\Omega}} \ .$$

The $\mathcal{N}=1$ chiral superfield $\phi^{(v)}$ (v for vector representation) is now transformed to $\phi^{(c)}$ in the chiral representation via the following change of basis:

$$\phi^{(v)} = e^{-\bar{\Omega}}\, \phi^{(c)}\, e^{\bar{\Omega}} \ , \quad \text{which gives} \ \ \bar{\phi}^{(v)} = e^{-\Omega}\, \bar{\phi}^{(c)}\, e^{\Omega} \ . \tag{15}$$

Plugging this into the action in (13), we obtain the more familiar form of the action for $\mathcal{N}=2$ SYM in $\mathcal{N}=1$ superspace:

$$\mathcal{S}_{\text{SYM}} = \frac{1}{32\pi} \text{Im} \int \mathrm{d}^4x\, \mathrm{d}^2\theta\ \text{tr} \left[\tau \left(W^A W_A - 4 \int \mathrm{d}^2\bar{\theta}\ e^{-V} \bar{\phi}\, e^V \phi \right) \right] \ , \tag{16}$$

where we have suppressed the superscript ${}^{(c)}$ on ϕ and $\bar{\phi}$. For later purposes, it is useful to write down the most general $\mathcal{N}=2$ SYM action in terms of the so-called *prepotential* $\mathcal{F}(W)$[2]:

$$\mathcal{S} = \frac{1}{16\pi} \text{Im} \int \mathrm{d}^4x\, \mathrm{d}^2\theta^1 \mathrm{d}^2\theta^2\, \mathcal{F}(W) \ , \tag{17}$$

with $\mathcal{F}(W) = \frac{1}{2}\, \tau\, W^i W_i$ reproducing the classical $\mathcal{N}=2$ action. In terms of $\mathcal{N}=1$ superspace, the action in the vector representation becomes

$$\mathcal{S} = -\frac{1}{32\pi} \text{Im} \int \mathrm{d}^4x\, \mathrm{d}^2\theta \left[\frac{\partial^2 \mathcal{F}(\phi)}{\partial\phi^i \partial\phi^j} \, (W^i)^A\, (W^j)_A - 4 \int \mathrm{d}^2\bar{\theta}\ (\phi^i)^\dagger\, \frac{\partial \mathcal{F}(\phi)}{\partial \phi^i} \right] \ . \tag{18}$$

[2]Superpotential would perhaps be a more logical name for $\mathcal{F}(W)$; the object Ω introduced in the previous paragraph is also referred to as the *prepotential* in the $\mathcal{N}=1$ superspace literature since it gives rise to the gauge potential A_A after expanding $\nabla_A = e^{-\Omega} D_A\, e^{\Omega}$.

We will see that more elaborate forms of $\mathcal{F}$ are needed to describe the low-energy effective action of $\mathcal{N}=2$ SYM theory. Looking at the coefficient of $(W^i)^A(W^j)_A$, we infer that there is a matrix of effective coupling constants given by

$$(\tau_{\rm eff})_{ij} = \frac{\partial^2 \mathcal{F}(\phi)}{\partial\phi^i\partial\phi^j}\ , \tag{19}$$

and note that, in particular, it depends on the field ϕ. The second term is the low-energy effective action for the Wess-Zumino multiplet, with the Kähler potential $K(\phi,\bar{\phi})$ taking the special form

$$K(\phi,\phi^\dagger) = \frac{1}{8\pi}\,{\rm Im}\ (\phi^i)^\dagger\,\frac{\partial\mathcal{F}(\phi)}{\partial\phi^i}\ . \tag{20}$$

A Kähler manifold with the above form of the Kähler potential is said to be *rigid special-Kähler.*

3 Low-energy effective theory on the Coulomb branch

3.1 The classical picture

The x-space action of $\mathcal{N}=2$ SYM with gauge group G is given by

$$\begin{aligned}\mathcal{L} = \frac{1}{g^2}{\rm tr}\Big[&\frac{1}{2}F_{\mu\nu}^2 + (D_\mu M)^2 + (D_\mu N)^2 + [M,N]^2 + \\ &- \bar{\lambda}_a\,\gamma^\mu D_\mu\lambda^a - i\bar{\lambda}_a\,[M\,,\lambda^a] - \bar{\lambda}_a\,\gamma_c[N\,,\lambda^a]\Big] - \frac{\theta}{16\pi^2}{\rm tr}\,F_{\mu\nu}{}^\star F^{\mu\nu}\ ,\end{aligned} \tag{21}$$

where λ^a, $a=1,2$ are the two Majorana spinors and M,N are the two spin-0 fields in the $\mathcal{N}=2$ vector multiplet. The Lie-algebra generators T_i are anti-hermitian and are in the fundamental representation with ${\rm tr}\,T_iT_j = -\frac{1}{2}\delta_{ij}$. The classical vacua are obtained by minimising the potential

$$V = -\frac{1}{g^2}{\rm tr}\,[M,N]^2\ .$$

In terms of the complex combination $\varphi = M+iN$, we see that the vacuum condition is $[\varphi,\varphi^\dagger]=0$. This implies that φ and $\varphi^\dagger$ must belong to the Cartan subalgebra of the gauge group G. A gauge invariant characterisation of such a φ is in terms of its eigenvalues. Once a basis is chosen for the Cartan generators H_k, gauge transformations preserving that basis serve to only permute the eigenvalues of φ. Since the H_k are traceless and are in the fundamental representation, φ has $N-1$ independent eigenvalues a_α, with $\alpha=1,\ldots,N-1$. Thus, there is an $N-1$ parameter space of flat directions for the potential. In other words, there is an $N-1$ dimensional moduli space of classical vacua, $\mathcal{M}_c$, parametrized by the a_α upto permutation. To characterise $\mathcal{M}_c$ in a completely gauge invariant fashion, we use the following manifestly gauge invariant objects:

$$\mathcal{M}_c:\quad U_k \equiv {\rm tr}\,\varphi^k\ ,\quad k=2,\ldots,N\ , \tag{22}$$

identifying $\mathcal{M}_c \simeq \mathbb{C}^{N-1}$ (U_1 is identically zero since φ is traceless.) We then treat the eigenvalues a_α as functions of the U_k. We can infer the following facts about the classical moduli space $\mathcal{M}_c$:

1. Different values of the U_k characterise *gauge-inequivalent* classical vacua. Since the vacuum energy is still zero, $\mathcal{N}=2$ supersymmetry is unbroken at all points on $\mathcal{M}_c$.

2. **The Coulomb branch:** At *generic* points in the moduli space (where no two eigenvalues of φ coincide), fluctuations about the vacuum are described by a $\mathcal{N}=2$ $U(1)^{N-1}$ gauge theory with $N(N-1)$ massive $\mathcal{N}=2$ BPS vector multiplets. These massive vector multiplets are charged under the $U(1)^{N-1}$ gauge group. In other words, there are $N-1$ "photons" and $N(N-1)$ "W-bosons". *For this reason, the moduli space of vacua is known as the Coulomb branch.*

3. **Singular loci:** Two eigenvalues (say a_1 and a_2) coincide on a complex $N-2$ dimensional surface in $\mathcal{M}_c$ since $a_1(U)=a_2(U)$ is one complex equation on the $N-1$ variables $\{U_k\}$. On this locus, two of the $N(N-1)$ W-bosons become massless and the gauge group is enhanced from $U(1)^{N-1}$ to $SU(2)\times U(1)^{N-2}$. For the case where more than two eigenvalues coincide, the gauge group is enhanced accordingly. All eigenvalues are equal only when they are all zero, and it follows that $U_k=0$ for all k. At this point all the W-bosons become massless and the full $SU(N)$ gauge group is restored.

We shall focus on the case of $SU(2)$ in what follows. In this case, there are two eigenvalues $a_1 \equiv a$ and $a_2 \equiv -a$. The classical moduli space space is parametrized by the single complex number $u = \operatorname{tr}\varphi^2 = -\frac{1}{2}a^2$. The gauge group is $U(1)$ away from $u=0$ and there is one $\mathcal{N}=2$ $U(1)$ multiplet and two massive, charged $\mathcal{N}=2$ BPS multiplets. At $u=0$ the W-bosons becomes massless and the full $SU(2)$ gauge group is restored.

3.2 Quantum corrections

We next look at the quantum case. Do the classically flat directions remain in the quantum theory? It is well known that perturbative corrections do not break supersymmetry in SYM theory. Also, it can be shown that there is no candidate for a superpotential term (which would lift the vacuum degeneracy) that is $\mathcal{N}=2$ supersymmetric. This means that supersymmetry cannot be broken non-perturbatively either. *Thus, the Coulomb branch persists in the quantum case.* However, the detailed structure of the theory (e.g. the singular loci) on the Coulomb branch might be potentially modified by quantum corrections. We thus call the quantum moduli space $\mathcal{M}_q$ to distinguish it from its classical counterpart. $\mathcal{M}_q$ is parametrized by the vacuum expectation values $u_k = \langle U_k\rangle = \langle \operatorname{tr}\varphi^k\rangle$.

Supersymmetry constrains the structure of various divergences that could occur in the quantum theory. In the present case of $\mathcal{N}=2$ supersymmetry it turns out that the perturbative β-function receives contributions only from one-loop diagrams and the contributions from two-loop and beyond are zero (This can be shown using superspace methods [HSW]. For an alternative argument by Seiberg using R-symmetry [Se], see Appendix A.). One can calculate the 1-loop β-function using conventional methods to obtain

$$\beta(g(\mu)) \equiv \mu \frac{\partial g}{\partial \mu} = -\frac{g(\mu)}{4\pi^2} \ . \tag{23}$$

where $g(\mu)$ is the running coupling constant and μ is the energy scale. We observe that our theory is asymptotically free and consequently that perturbation theory is valid only at very high energies.

Integrating the above equation between the energy scales μ and $\widetilde{\mu}$, we see that the following quantity is invariant under renormalization group flow:

$$\frac{4\pi}{g(\widetilde{\mu})^2} - \frac{1}{\pi}\log\frac{\widetilde{\mu}^2}{\Lambda^2} = \frac{4\pi}{g(\mu)^2} - \frac{1}{\pi}\log\frac{\mu^2}{\Lambda^2} = \frac{1}{\pi}\log c \ , \tag{24}$$

where Λ is a constant with the dimension of mass and c is a numerical constant, both independent of the scale μ. Λ is then given by

$$\Lambda^2 = c\mu^2 \exp\left(-\frac{4\pi^2}{g(\mu)^2}\right) \ , \tag{25}$$

which implies that when $\mu \sim \Lambda$, the coupling constant becomes very large. Λ is also invariant under RG flow and is a dynamically generated energy scale at which the running coupling becomes large. Since the β-function is 1-loop exact, one can complexify the running coupling $g(\mu)$ in the usual way to $\tau(\mu)$ to get

$$\tau(\mu) = \frac{i}{\pi}\log\frac{c\mu^2}{\Lambda^2} \ . \tag{26}$$

(If there were higher loop corrections, they would be proportional to powers of $\operatorname{Im}\tau$, and the β-function could no longer have been complexified. The holomorphic dependence of the β-function on τ would then be lost.) The microscopic θ-angle in $\tau(\mu)$ does not run in perturbation theory and hence can be absorbed into Λ as a phase:

$$\Lambda^2 = c\mu^2 \exp\left(\pi i \tau(\mu)\right) = c\mu^2 \exp\left(-\frac{4\pi^2}{g(\mu)^2} + \frac{i\theta}{2}\right) \ . \tag{27}$$

3.3 Low-energy effective theory

Henceforth, we shall be primarily interested in the low-energy effective theory. We saw that perturbation theory does not lift the classically flat directions and hence there is a quantum moduli space of vacua $\mathcal{M}_q$ parametrized by the expectation value u. What is the low-energy behaviour at different points of $\mathcal{M}_q$?

Classically, we had $U_2 = \operatorname{tr}\varphi^2 = -a^2/2$. In quantum theory, this continues to hold for $u = \langle U_2 \rangle$ at high energies where perturbation theory is valid. Indeed,

$$\text{High energy, } \mu \to \infty: \quad u = -\frac{1}{2}a^2 + \text{small } \mathcal{O}(g(\mu)) \text{ corrections} . \tag{28}$$

Also, since the semiclassical analysis is valid in this regime, we can conclude that quantum fluctuations are described by a $\mathcal{N} = 2$ $U(1)$ gauge theory with two oppositely charged W-bosons whose masses are proportional to $|a|$.

Now, if we lower the energy scale μ to energies below $|a|$, we have to integrate out the W-bosons *a la* Wilson to obtain an effective action for fluctuations with energy up to $\mu = |a|$. This is then described by a $\mathcal{N} = 2$ $U(1)$ gauge multiplet which consists only of $U(1)$-*neutral* x-space fields. Consequently, the β-function becomes zero below $\mu \sim |a|$ and the coupling $\tau(\mu)$ stops running. If $|a| \gg |\Lambda|$, the coupling stops running before it becomes too large and hence the low-energy effective theory is weakly coupled. Since $u = -a^2/2$ in this regime, the low-energy effective theory is weakly coupled $\mathcal{N} = 2$ $U(1)$ gauge theory for $|u| \gg |\Lambda|^2$, i.e. in the neighbourhood of $u \to \infty$ in the moduli space $\mathcal{M}_q$.

Since the flat directions persist after including perturbative quantum corrections, the low-energy fluctuations can be described by giving the modulus a a slowly-varying spacetime dependence with its value fixed at the constant a at asymptotic infinity. In other words, the low-energy effective theory contains a scalar $A(x)$ with $\langle A(x) \rangle = a$. Since the theory is $\mathcal{N} = 2$ supersymmetric, this slowly-varying scalar $A(x)$ lies in a $\mathcal{N} = 2$ chiral multiplet described by a superfield $\mathcal{A}$ with the following $\mathcal{N} = 1$ content:

$$\mathcal{A}_| \equiv A \ , \qquad \nabla_{2A}\, \mathcal{A}_| \equiv W_A \ , \tag{29}$$

The $\mathcal{N} = 1$ chiral superfield which contains the scalar A is denoted by the same letter and we have used the same symbol W_A from earlier to denote the $\mathcal{N} = 1$ $U(1)$ field strength to stick to standard usage. The coupling constant of the low energy theory is frozen at its value at $\mu = |a|$ and is given by

$$\tau(a) = \frac{i}{\pi} \log \frac{c|a|^2}{\Lambda^2} \ . \tag{30}$$

Such a coupling is generated by the following $\mathcal{N} = 2$ prepotential for the $U(1)$ gauge theory described by $\mathcal{A}$:

$$\mathcal{F}(\mathcal{A}) = \frac{i}{2\pi}\, \mathcal{A}^2 \log\left(\frac{\mathcal{A}^2}{\Lambda^2}\right) \ , \tag{31}$$

and the low-energy effective action (with at most two derivatives) takes the following form:

$$\mathcal{S} = -\frac{1}{32\pi} \mathrm{Im} \int \mathrm{d}^4x\, \mathrm{d}^2\theta \left[\frac{\partial^2 \mathcal{F}(A)}{\partial A^2}\, W^A W_A - 4 \int \mathrm{d}^2\bar{\theta}\, A^\dagger\, \frac{\partial \mathcal{F}(A)}{\partial A} \right] \ . \tag{32}$$

This is a non-linear sigma model for the scalar A with the moduli space $\mathcal{M}_q$ as target space. The fluctuations in A drive fluctuations of the gauge field and of the various fermions in the supersymmetry multiplet. The effective coupling constant is

$$\tau_{\text{eff}}(A) = \frac{\partial^2 \mathcal{F}(A)}{\partial A^2} = \frac{i}{\pi}\left(\log\frac{A^2}{\Lambda^2}+3\right) \xrightarrow[x\to\infty]{} \frac{i}{\pi}\left(\log\frac{a^2}{\Lambda^2}+3\right) . \tag{33}$$

We see that the low-energy effective action gives $c = e^3$. From now on, we will absorb this constant into Λ. Comparing this with the τ in (30), we see that a, rather than $|a|$, occurs in τ_{eff}. This is due to the requirement that $\mathcal{F}(\mathcal{A})$ be holomorphic in $\mathcal{A}$. This simple fact leads to the generation of an effective θ-angle at low energies, given by

$$\frac{\theta(a)_{\text{eff}}}{2\pi} = \frac{\theta}{2\pi} - \frac{2\arg(a)}{\pi} . \tag{34}$$

What is the description of the vacuum moduli space, $\mathcal{M}_q$, in terms of the low-energy variables? In the microscopic theory, $\mathcal{M}_q$ was described in terms of the the gauge-invariant vacuum expectation value $u = \langle \operatorname{tr}\varphi^2\rangle$. In terms of low-energy variables u is given by $u = -\langle A^2\rangle/2$. In the regime $u \to \infty$ where the low-energy theory is weakly coupled, one can calculate u reliably using perturbation theory and get $u = -a^2/2$, the same as in the microscopic description.

Summarising, the low-energy effective theory is weakly coupled $\mathcal{N} = 2$ U(1) gauge theory without additional matter in the neighbourhood of $u \to \infty$ in $\mathcal{M}_q$. In this regime $u = -a^2/2$ and hence a is also a good coordinate on the moduli space $\mathcal{M}_q$ in this neighbourhood. As we go to smaller values of u, perturbation theory ceases to be reliable since the low-energy coupling becomes stronger. Then, it is no longer clear what the low-energy effective description should be, apart from the fact that it is $\mathcal{N} = 2$ supersymmetric.

To explore this region of the moduli space, we should first take into account the non-perturbative quantum corrections which we have ignored so far. Such effects are extremely small in the semiclassical region $u \to \infty$. But as u become smaller, they become more important and have to be included in any sensible treatment of the theory for small u. We discuss this next.

4 Non-perturbative effects

Instantons in $SU(2)$ Yang-Mills theory are known to provide non-perturbative quantum corrections to the effective action. In the high energy regime where the theory is weakly coupled, the contribution to the path integral from the perturbative effective action evaluated on a k-instanton solution is given by

$$\exp\left(-\frac{8\pi^2 k}{g(\mu)^2}+ik\theta\right)=\exp\left(2\pi i k\tau(\mu)\right)=\left(\frac{\Lambda}{\mu}\right)^{4k}\ , \tag{35}$$

where we have used the formula in (27) for Λ. We observe that this is indeed beyond all orders in perturbation theory for small $g(\mu)$. After including perturbative corrections about such a background, we can exponentiate back the value of the path integral and include it in the effective action. These will then be interpreted as non-perturbative corrections to the effective action. In $\mathcal{N}=2$ language, the prepotential is expected to get non-perturbative corrections of the following form from the k-instanton sector:

$$\Delta^{(k)}\mathcal{F}(\mathcal{A})=\frac{1}{2\pi i}\,f_k(\mathcal{A})\left(\frac{\Lambda}{\mathcal{A}}\right)^{4k}\ , \tag{36}$$

where $f_k(\mathcal{A})$ denotes the contribution from perturbation theory about the k-instanton background that we have not calculated yet. The $1/2\pi$ is a convenient normalisation and the i comes from Wick rotating the effective action back from Euclidean to Minkowski space.

4.1 R-symmetry

In order to get a handle on the precise form of the instanton contributions, we first study the $U(1)_{\mathrm{R}}$ chiral R-symmetry that is violated in an instanton background. This R-symmetry acts on the supercoordinates as $\theta\to e^{i\alpha}\theta$ and $\bar\theta\to e^{-i\alpha}\bar\theta$. The classical x-space Lagrangian $\int \mathrm{d}^2\theta^1\mathrm{d}^2\theta^2\,\frac{1}{2}\tau_{\mathrm{cl}}\mathcal{A}^2$ is expected to be invariant under this R-symmetry. This is achieved if $\mathcal{A}$ transforms as

$$\mathcal{A}(x,\theta,\bar\theta)\to e^{2i\alpha}\mathcal{A}(x,e^{i\alpha}\theta,e^{-i\alpha}\bar\theta)\ . \tag{37}$$

Next, we study how the perturbative prepotential transforms under $U(1)_{\mathrm{R}}$. We have

$$\mathcal{F}(\mathcal{A})_{\mathrm{pert}}=\frac{i}{2\pi}\mathcal{A}^2\log\frac{\mathcal{A}^2}{\Lambda^2}\to e^{4i\alpha}\left(\mathcal{F}(\mathcal{A})_{\mathrm{pert}}-\frac{2\alpha}{\pi}\mathcal{A}^2\right)\ . \tag{38}$$

The effective Lagrangian density $\mathcal{L}_{\mathrm{eff}}=\frac{1}{16\pi}\int \mathrm{d}^2\theta^1\mathrm{d}^2\theta^2\,\frac{1}{2}\tau(\mathcal{A})\mathcal{A}^2$ then transforms as

$$\mathcal{L}_{\mathrm{eff}}\to\mathcal{L}_{\mathrm{eff}}-\frac{\alpha}{4\pi^2}F_{\mu\nu}{}^{\star}F^{\mu\nu}\ , \tag{39}$$

where $F_{\mu\nu}$ is the low-energy $U(1)$ field strength. The effective Lagrangian is *not invariant* under $U(1)_{\mathrm{R}}$, signalling an anomaly. In fact, since the $U(1)_{\mathrm{R}}$ symmetry is a chiral symmetry, one expects an anomaly. The above result (39) can be obtained by evaluating a 1-loop triangle graph with one axial and two vector currents on the external legs and chiral fermions running in the loop. Then one observes that the divergence of the axial current corresponding to the rigid $U(1)_{\mathrm{R}}$ symmetry is not zero and is the right hand side of (39).

The change in $\mathcal{L}_{\text{eff}}$ in (39) can be compensated by a shift in the microscopic θ-angle of the theory:

$$\theta \rightarrow \theta + 8\alpha \ . \tag{40}$$

Since a change in θ by (2π times) an integer does not change the physics, we see that a discrete subgroup of $U(1)_{\text{R}}$ with $e^{i\alpha} \in \mathbb{Z}_8$ is actually still a symmetry of the theory. This is true even after including instanton effects since the θ-angle dependence of the k-instanton contribution is $e^{ik\theta}$, and the $\mathbb{Z}_8$ shifts θ by an integer multiple of 2π. Alternatively, one can start with the instanton action and infer that only a $\mathbb{Z}_8$ subgroup of $U(1)_{\text{R}}$ survives in the instanton background.

The vacuum is characterised by the expectation value $u = \langle \operatorname{tr} \varphi^2 \rangle$. Since $\varphi \rightarrow e^{2i\alpha}\varphi$ under $U(1)_{\text{R}}$, we have $u \rightarrow e^{4i\alpha}u$. Hence the vacuum characterised by u is invariant only under a $\mathbb{Z}_4$ subgroup of the $\mathbb{Z}_8$ R-symmetry. We thus have the following pattern:

$$U(1)_{\text{R}} \xrightarrow[\text{corrections}]{\text{1-loop}} \mathbb{Z}_8 \xrightarrow[\text{broken}]{\text{spontaneously}} \mathbb{Z}_4 \ . \tag{41}$$

The remaining $\mathbb{Z}_2 \subset \mathbb{Z}_8$ acts on u as a spontaneously broken symmetry

$$u \longrightarrow -u \ . \tag{42}$$

4.2 Non-perturbative corrections to the prepotential

There is another, more powerful way of interpreting the anomalous shift of the θ-angle. We recall that the microscopic θ-angle was absorbed into the strong coupling scale as a phase, with

$$\Lambda^2 = \mu^2 \exp\left(-\frac{4\pi^2}{g(\mu)^2} + \frac{i\theta}{2} \right) \ . \tag{43}$$

The shift $\theta \rightarrow \theta + 8\alpha$ then translates into the following transformation of Λ:

$$\Lambda \rightarrow e^{2i\alpha}\Lambda \ , \tag{44}$$

This says that the anomaly effectively gives an R-weight of 2 to Λ. Seiberg's insight in the early 1990s was that Λ can be treated as an actual $\mathcal{N}=1$ chiral superfield that transforms with R-weight 2 since it appears holomorphically in the prepotential. This heavily constrains the form of possible corrections to the prepotential in $\mathcal{N}=2$ theories, and more generally, the superpotential in $\mathcal{N}=1$ theories.

If one assigns an R-weight of 2 to Λ, the $U(1)_{\text{R}}$ symmetry is restored and the prepotential in the full quantum theory must transform homogeneously with R-weight 4. This implies that the precise form of the k-instanton contribution is

$$\Delta^{(k)}\mathcal{F}(\mathcal{A}) = \frac{1}{2\pi i}\mathcal{F}_k\, \mathcal{A}^2 \left(\frac{\Lambda}{\mathcal{A}} \right)^{4k} \tag{45}$$

where $\mathcal{F}_k$ is a constant to be determined. Thus, using simple R-symmetry arguments, we have inferred the exact form of the instanton contribution to the prepotential. The constants $\mathcal{F}_k$ cannot be determined this way. Thus, the most general form of the prepotential $\mathcal{F}(\mathcal{A})$ that is well-defined in the weak-coupling limit $a \to \infty$ is given by

$$\mathcal{F}(\mathcal{A}) = \frac{i}{2\pi}\,\mathcal{A}^2 \log\left(\frac{\mathcal{A}^2}{\Lambda^2}\right) + \frac{1}{2\pi i}\sum_{k=1}^{\infty} \mathcal{F}_k\,\mathcal{A}^2\left(\frac{\Lambda}{\mathcal{A}}\right)^{4k} , \tag{46}$$

This was first written down by Seiberg in [Se]. The contributions from instantons with $k < 0$ are absent because terms with $k < 0$ blow up as $a \to \infty$, which contradicts with the existence of a well-defined semiclassical limit. Without a knowledge of the $\mathcal{F}_k$, we cannot say whether the above series is convergent. However, it is an asymptotic series as $a \to \infty$ since each term in the instanton series is sub-leading compared to the previous one.

From the above formula for the prepotential, we see that as one goes away from $u = \infty$ in the moduli space, the non-perturbative terms become large. Thus, on top of perturbation theory breaking down, we have an infinite number of large non-perturbative corrections to worry about. Thus the description of the low-energy theory in terms of the chiral multiplet $\mathcal{A}$ seems to be in trouble for smaller values of u.

There is yet another issue with the description of the low-energy effective theory in terms of $\mathcal{A}$ for smaller values of u which we next discuss. On the face of it, this seems to compound our troubles, but it turns out that this issue contains the germ for a resolution as well.

5 Duality

5.1 Introducing the variable $a_D(u)$

The low-energy effective theory for the scalar A in the neighbourhood $u \to \infty$ is described by a non-linear sigma model with $\mathcal{M}_q$ as target. The Kähler metric on this target space (see end of Section 1) is given by

$$\mathrm{d}s^2 = \frac{1}{8\pi}\mathrm{Im}\,\mathcal{F}''(a)\,\mathrm{d}a\,\mathrm{d}\bar{a}\ . \tag{47}$$

The coefficient of the kinetic energy for A is then proportional to $\mathrm{Im}\,\mathcal{F}''(A)$. Since $\mathrm{Im}\,\mathcal{F}'' = \frac{4\pi}{g(a)^2}$ is positive, the coefficient of kinetic energy is positive as is necessary for a unitary theory.

However, we run into trouble for smaller values of a. As a decreases, $g(a)$ increases and $\mathrm{Im}\,\mathcal{F}''$ might hit a zero and even become negative! In fact, this is required by holomorphy of $\mathcal{F}(a)$. The holomorphic nature of $\mathcal{F}(a)$ implies that $\mathrm{Im}\,\mathcal{F}''(a)$ is a harmonic function and hence it cannot be positive in the entire a plane (since otherwise it would have a minimum in its domain of definition). This implies that the sign of the kinetic energy has to be negative in some region in the a-plane which is bad news for unitarity.

The only way out is to conclude that a cannot be a good coordinate on the whole moduli space. Suppose there is a global coordinate u (e.g. $u = \langle \mathrm{tr}\, \varphi^2 \rangle$). Then, the metric can be expressed in terms of u as follows:

$$\mathrm{d}s^2 = \mathrm{Im} \frac{\mathrm{d}a_D}{\mathrm{d}u} \frac{\mathrm{d}\bar{a}}{\mathrm{d}\bar{u}} \,\mathrm{d}u\, \mathrm{d}\bar{u} = \frac{1}{2i} \left(\frac{\mathrm{d}a_D}{\mathrm{d}u} \frac{\mathrm{d}\bar{a}}{\mathrm{d}\bar{u}} - \frac{\mathrm{d}a}{\mathrm{d}u} \frac{\mathrm{d}\bar{a}_D}{\mathrm{d}\bar{u}} \right) \mathrm{d}u\, \mathrm{d}\bar{u} \tag{48}$$

where we have defined $a_D = \mathcal{F}'(a)$. We also define, for later purposes, $A_D = \mathcal{F}'(A)$. The metric in this form is valid globally on the moduli space. However, the positivity of the metric is still not clear from this description. This will be fixed later.

The variables a and a_D are treated on an equal footing here. Writing (a_D, a) as a column vector x, it is easy to see that the form of the Kähler metric above is preserved by transformations of the form

$$x \longrightarrow Mx + c \ , \quad \text{with} \quad c \in \mathbb{C}^2 \ \text{ and } \ M \in SL(2,\mathbb{R}) \ . \tag{49}$$

Later, we shall see that constant shifts of x are not allowed i.e. c has to be zero. The group $SL(2,\mathbb{R})$ is generated by

$$T_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \ , \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ . \tag{50}$$

The action of T_b is to shift $a_D \to a_D + ba$, which results in a shift of the θ-angle by $2\pi b$ since $\tau(a) = \mathrm{d}a_D/\mathrm{d}a$ transforms as $\tau(a) \to \tau(a) + b$. Since the physics is invariant only under (2π times) integer shifts of the θ-angle, b must be an integer. This reduces $SL(2,\mathbb{R})$ to its discrete subgroup $SL(2,\mathbb{Z})$ which is generated by $T_{b=1}$(henceforth called T) and S.

5.2 Electric-magnetic duality

Next, let us look at the action of S. We have

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \xrightarrow{\ S\ } \begin{pmatrix} a \\ -a_D \end{pmatrix} \ . \tag{51}$$

The $\mathcal{N} = 1$ action for the superfield A can be recast into an action for A_D by introducing a second function $\mathcal{F}_D(A_D)$ satisfying $\mathcal{F}_D'(A_D) = -A$:

$$\begin{aligned} \mathcal{S}[A] &= -\frac{1}{8\pi} \mathrm{Im} \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta} \left[\bar{A}\, \mathcal{F}'(A) \right] \ , \\ &= \frac{1}{8\pi} \mathrm{Im} \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta} \left[\overline{\mathcal{F}_D'(A_D)}\, A_D \right] = -\frac{1}{8\pi} \mathrm{Im} \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta} \left[\bar{A}_D\, \mathcal{F}_D'(A_D) \right] \ . \end{aligned} \tag{52}$$

where the primes indicate derivatives with respect to the appropriate argument. Using $A_D = \mathcal{F}'(A)$ and $A = -\mathcal{F}_D'(A_D)$, we see that $\mathcal{F}_D$ and $\mathcal{F}$ are Legendre transforms of each other:

$$\mathcal{F}_D(A_D) = \mathcal{F}(A) - A\,A_D\;, \tag{53}$$

and also that

$$\mathcal{F}_D''(A_D) = -\frac{\mathrm{d}A}{\mathrm{d}A_D} = -\frac{1}{\mathcal{F}''(A)}\;. \tag{54}$$

One can elevate the function $\mathcal{F}_D(A_D)$ to a genuine $\mathcal{N}=2$ prepotential by considering an $\mathcal{N}=2$ chiral superfield $\mathcal{A}_D$ whose $\theta^2 = \bar{\theta}^2 = 0$ component is A_D. What is the physical interpretation of $\mathcal{A}_D$?

The $\mathcal{N}=1$ chiral field strength W_A satisfies the reality constraint

$$\nabla^A W_A = \overline{\nabla}^{\dot{A}}\overline{W}_{\dot{A}}\;, \quad \text{that is} \quad \mathrm{Im}\,(\nabla^A W_A) = 0\;. \tag{55}$$

We could impose this constraint in the action by first treating W_A as an independent chiral field and then adding a Lagrange multiplier term, with Lagrange multiplier $\widetilde{V}$ which is a real $\mathcal{N}=1$ superfield:

$$\mathcal{S}[W_A,\widetilde{V}] = \frac{1}{32\pi}\mathrm{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\left[\mathcal{F}''(A)W^AW_A + 8\int \mathrm{d}^2\bar{\theta}\;\widetilde{V}\,\nabla^A W_A\right] \tag{56}$$

Now, we partially integrate the gauge covariant derivative ∇^A so that it hits $\widetilde{V}$ and use $-4\int \mathrm{d}^2\bar{\theta} = \overline{\nabla}^2$ upto total derivatives. We then write $\widetilde{W}^A = \overline{\nabla}^2\nabla^A\widetilde{V}$ to get

$$\mathcal{S}[W_A,\widetilde{V}] = \frac{1}{32\pi}\mathrm{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\left[\mathcal{F}''(A)W^AW_A - 2\,\widetilde{W}^A\,W_A\right]\;. \tag{57}$$

We can now integrate out the independent chiral superfield W_A to obtain an action for $\widetilde{W}_A$:

$$\begin{aligned}\mathcal{S}[\widetilde{W}] &= \frac{1}{32\pi}\mathrm{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\left[\frac{-1}{\mathcal{F}''(A)}\widetilde{W}^A\,\widetilde{W}_A\right]\\ &= \frac{1}{32\pi}\mathrm{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\left[\mathcal{F}_D''(A_D)\widetilde{W}^A\,\widetilde{W}_A\right]\;.\end{aligned} \tag{58}$$

This is the $\mathcal{N}=1$ analog of the standard electric-magnetic duality transformation where we introduce a *dual* gauge field $(A_D)_\mu$ that couples to the Bianchi identity $\partial^\mu {}^\star F_{\mu\nu} = 0$ and integrate out the original field strength in favour of the dual field strength.

Combining this with (52), we see that, in terms of $\widetilde{W}$ and A_D, the original $\mathcal{N}=2$ action becomes

$$\mathcal{S}[\widetilde{W},A_D] = \frac{1}{32\pi}\mathrm{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\left[\mathcal{F}_D''(A_D)\widetilde{W}^A\widetilde{W}_A - 4\int \mathrm{d}^2\bar{\theta}\;\bar{A}_D\mathcal{F}_D'(A_D)\right]\;, \tag{59}$$

Thus, $\widetilde{W}_A$ couples to the $\mathcal{N}=1$ chiral superfield A_D as the $(\nabla_{2A}\mathcal{A}_D)_|$ component in the $\mathcal{N}=2$ chiral multiplet $\mathcal{A}_D$. That is, $\widetilde{W}_A$ and A_D form an $\mathcal{N}=2$ chiral multiplet $\mathcal{A}_D$ that is the magnetic dual of the chiral multiplet $\mathcal{A}$.

The upshot of this calculation is that S acting on (a_D, a) implements electric-magnetic duality on the $U(1)$ gauge fields and also inverts the complex coupling constant:

$$S: \quad \text{electric} \leftrightarrow \text{magnetic} \ , \quad \mathcal{A} \leftrightarrow \mathcal{A}_D \ , \quad \tau \leftrightarrow -\frac{1}{\tau} \ . \tag{60}$$

Crucially, this also suggests a solution to our strong coupling problem. In the regions of moduli space where the original $U(1)$ gauge theory strongly coupled (small $\mathrm{Im}\,\tau$), one can perform an S-transformation to the dual description which is weakly coupled (large $\mathrm{Im}\,\tau$)! In fact, there exist infinitely many different weakly coupled descriptions of the original $U(1)$ theory related to it by $SL(2,\mathbb{Z})$ transformations. The hope then is that, perhaps one (or more) of these descriptions is (are) apt for the finite region of the u-plane.

6 Interlude

Let us take stock of where we stand in our analysis. The backbone of the entire program is $\mathcal{N}=2$ supersymmetry. Due to $\mathcal{N}=2$ supersymmetry and standard arguments involving asymptotic freedom and anomalies in chiral symmetries, we have

1. shown that there is a non-trivial quantum moduli space of vacua $\mathcal{M}_q$ parametrized by $u \in \mathbb{C}$,

2. argued that the low-energy effective theory is weakly coupled $U(1)$ gauge theory for large values of $|u|$ and that $a = \sqrt{-2u}$ is a good coordinate on moduli space in the neighbourhood of $u \to \infty$ and

3. obtained the exact form of the $\mathcal{N}=2$ prepotential of the low-energy effective theory for large $|u|$:

$$\mathcal{F}(\mathcal{A}) = \frac{i}{2\pi}\,\mathcal{A}^2 \log\left(\frac{\mathcal{A}^2}{\Lambda^2}\right) + \frac{1}{2\pi i}\sum_{k=1}^{\infty}\mathcal{F}_k\,\mathcal{A}^2\left(\frac{\Lambda}{\mathcal{A}}\right)^{4k} \ . \tag{61}$$

The outstanding issues are

1. The low-energy effective theory becomes strongly coupled for smaller values of a and The instanton expansion in $\mathcal{F}(\mathcal{A})$ is no longer sub-leading,

2. The kinetic energy for the non-linear sigma model fails to remain positive for smaller values of a as a consequence of the holomorphic nature of $\mathcal{F}(\mathcal{A})$.

These issues led us to conclude that a, a physical parameter proportional to the masses of the W-bosons, should not a good coordinate on the moduli space $\mathcal{M}_q$. A way out of these problems was suggested (based on the arguments in [SW1]) in the previous section by introducing an extra variable $a_D = \mathcal{F}'(a)$ and expressing the Kähler metric in terms of a global coordinate u:

$$\mathrm{d}s^2 = \frac{1}{2i}\left(\frac{\mathrm{d}a_D}{\mathrm{d}u}\frac{\mathrm{d}\bar{a}}{\mathrm{d}\bar{u}} - \frac{\mathrm{d}a}{\mathrm{d}u}\frac{\mathrm{d}\bar{a}_D}{\mathrm{d}\bar{u}}\right)\mathrm{d}u\,\mathrm{d}\bar{u}\ , \tag{62}$$

in terms of two locally well-defined functions $a(u)$ and $a_D(u)$ on the u-plane. The form of the metric above is preserved by $SL(2,\mathbb{Z})$ transformations acting on (a_D, a).

Plan: The next step in our analysis will be to determine the functions $a(u)$ and $a_D(u)$ on the u-plane. Once we know a and a_D, we can determine the Kähler metric with the above formula. More importantly, we can integrate $a_D = \mathcal{F}'(a)$ to obtain $\mathcal{F}$ as a function of u! The integration constants can then be fixed using the known form of the prepotential for $u \to \infty$.

Thus, solving for the low-energy effective theory of the model is equivalent to determining the functions $a(u)$ and $a_D(u)$.

7 Low-energy dynamics at strong coupling

In this section, we get to work. Using the complex-analytic properties of the prepotential and its derivatives, we progress into the finite region of the u plane where the low-energy coupling becomes strong.

We choose the global coordinate on $\mathcal{M}_q$ to be $u = \langle \mathrm{tr}\,\varphi^2\rangle$. What happens to the objects $a_D(u)$ and $a(u)$ as we take u in a small circle around ∞? In this region, $\mathcal{F}$ is well-approximated by its 1-loop term and a and a_D are given in terms of u as

$$\begin{aligned} a(u) &\sim \sqrt{-2u}\ , \\ a_D(u) &= \frac{\partial \mathcal{F}(a)}{\partial a} \sim \frac{i\sqrt{-2u}}{\pi}\left(2\log\frac{\sqrt{-2u}}{\Lambda} + 1\right)\ . \end{aligned} \tag{63}$$

Hence, when $u \to ue^{2\pi i}$, we have $a \to e^{i\pi}a = -a$ and $a_D \to -a_D + 2a$. Thus, there is a monodromy M_∞ around $u = \infty$ given by

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_D \\ a \end{pmatrix} = PT^{-2}\begin{pmatrix} a_D \\ a \end{pmatrix}\ . \tag{64}$$

First we note that the monodromy matrix has to be an element of $SL(2,\mathbb{Z})$ since we determined the biggest group of transformations which preserve the Kähler structure on $\mathcal{M}_q$ to be $SL(2,\mathbb{Z})$. Secondly, the factor P acts as $a \to -a$ and $a_D \to -a_D$. This is simply the permutation of the eigenvalues a and $-a$ of φ extended to a_D (Classically, $a_D = \tau_{\rm cl} a$ and hence under $a_D \to -a_D$ when $a \to -a$). The factor T^{-2} is a purely quantum effect. We will see later that it is related to the Witten effect [Wi]: In the presence of a non-zero θ-angle, a positively charged magnetic monopole gets an electric charge of $\theta/2\pi$.

The monodromy analysis is what will help us make inroads into the finite region of the u-plane. Since ∞ is a branch point, there has to be atleast another branch point in the finite u-plane where the branch cut starting from ∞ has to end.

One branch point in the finite u-plane:
Let us assume that there is only one branch point in the finite u-plane. Then, using the $u \to -u$ R-symmetry, we see that it has to be at $u = 0$. Since we have assumed only one branch point in the finite u-plane, we can "lasso" the circle that winds around $u = \infty$ all the way down to $u = 0$ without changing the monodromy. Then, the monodromy M_0 around $u = 0$ has to be the same as M_∞:

$$M_0 = M_\infty \ . \tag{65}$$

(The monodromy matrix is always written such that it corresponds to the *counterclockwise* contour around the branch point). This means that $u = -a^2/2$ is valid at $u = 0$ as well, and hence the relation is valid throughout the u-plane, something which we ruled out based on unitarity. Hence, our assumption that there are only two branch points in the u-plane is incorrect. Also, this implies that the classical singularity at $u = 0$ is not present anymore in $\mathcal{M}_q$.

Two branch points in the finite u-plane:
The next simplest thing to try is a pair of branch points in the finite u-plane, related to each other by $u \to -u$. Let the branch points be at $u = \pm u_0 \neq 0$. Then, by deforming the counterclockwise circular contour around $u = \infty$ such that it is now tightly wound counterclockwise first around $u = -u_0$ and then around $u = u_0$, we see that the corresponding monodromies have to satisfy

$$M_{u_0} M_{-u_0} = M_\infty \ . \tag{66}$$

To proceed further, we need more information regarding the physics at the points u_0 and $-u_0$. What are the possibilities?

Massless W-bosons?
The first thought is that the classical singularity at $u = 0$ where the W-bosons become massless has now shifted to one of $\pm u_0$. However, this is implausible for several reasons. On general grounds, one expects a theory which has massless gauge bosons in the infrared to be conformal. Combined with $\mathcal{N} = 2$ supersymmetry, the effective action of the low-energy theory has to be invariant under the full $\mathcal{N} = 2$ superconformal symmetry. This rules out an anomaly in $U(1)_{\rm R}$ since it is now a part of the superconformal algebra. So, the anomaly

in R-symmetry has to vanish somehow at this point. Also, the non-zero expectation value $\langle \operatorname{tr} \varphi^2 \rangle = \pm u_0 \neq 0$ is in conflict with the fact that there are no preferred energy scales in a conformally invariant theory.

So, we must go with a different guess. There are no other elementary particles in the spectrum of the theory. There exist, however, stable solitonic solutions which are extremely heavy and classical in the weakly-coupled regime $u \to \infty$. These are the magnetic monopoles. What happens to these solitonic excitations as we go to smaller values of $|u|$? To answer this question, we must study the masses of these excitations as a function of the moduli space coordinates. This is what we do next.

7.1 Central charge and the BPS formula

We first observe that since the theory is $\mathcal{N} = 2$ supersymmetric, all the excitations must organise themselves into multiplets under the supersymmetry. In particular, a massive multiplet with maximal spin $j = 1/2$ has to be a hypermultiplet which saturates the BPS bound

$$M = |Z| \ . \tag{67}$$

Thus, the magnetic monopole must belong to a BPS hypermultiplet.

Next, we recall (from Homework 11) that, in the classical theory, the real and imaginary parts of the central charge are proportional to the electric charge Q_e and magnetic charge Q_m measured at asymptotic infinity (this was first demonstrated by Olive and Witten in [WO]). With the normalisation of the gauge kinetic term adopted here, the electric charge Q_e is integer-valued, $Q_e = n_e$, whereas the magnetic charge is integer-valued in units of $4\pi/g^2$, $Q_m = 4\pi n_m/g^2$. Thus the central charge is

$$Z = -a\left(n_e + \frac{4\pi i}{g^2} n_m\right) \ . \tag{68}$$

Once a θ-angle is turned on, n_e gets shifted by $\theta/2\pi$ times the magnetic quantum number n_m. This is known as the Witten effect [Wi](See the references for a derivation). Including this contribution, the formula for the central charge becomes

$$\operatorname{Re} Z = -a\left(n_e + \frac{\theta}{2\pi} n_m\right) \ , \quad \operatorname{Im} Z = -a\,\frac{4\pi}{g^2}\, n_m \ . \tag{69}$$

In other words,

$$Z = -a(n_e + \tau_{\rm cl} n_m) \ . \tag{70}$$

What is the analog of the above formula in the quantum theory? Firstly, the BPS relation $M = |Z|$ holds in the quantum theory as well, since it is a consequence of the $\mathcal{N} = 2$

supersymmetry algebra. However, the formula for Z in terms of the conserved electric and magnetic charges must receive quantum corrections. A simple way to guess the correct quantum-corrected formula for Z is to use the duality transformation that switches electric and magnetic variables. Under the S-transformation we have

$$\begin{aligned} a \longrightarrow -a_D \ , &\quad a_D \longrightarrow a \ , \\ F_{\mu\nu} \longrightarrow -{}^\star F_{\mu\nu} \ , &\quad {}^\star F_{\mu\nu} \longrightarrow F_{\mu\nu} \ , \\ n_e \longrightarrow -n_m \ , &\quad n_m \longrightarrow n_e \ . \end{aligned} \tag{71}$$

Let us consider the theory for large u, where the semiclassical approximation is valid. The contribution to the central charge due to a purely electrically charged state is then $Z = -a n_e$. Under the above duality transformation, this maps to $Z = -a_D n_m$. Thus, the contribution to the central charge due to a magnetic monopole should be $-a_D n_m$ in the semi-classical regime. Since the central charge is additive, the total central charge for a state with electric charge n_e and magnetic charge n_m must be given by

$$\boxed{Z = -a n_e - a_D n_m \ .} \tag{72}$$

Now, the above formula is in terms of a and a_D which are renormalization group invariant. Thus, this formula is valid even away from the $u \to \infty$ region in moduli space! Also, we see that the above formula reduces to (70) in the classical limit.

Duality invariance: Via the BPS relation $M = |Z|$, (72) relates the masses of the particles to their electric and magnetic charges. Since particle masses should not depend on which description we use for the low-energy theory, the BPS formula implies that the central charge formula (72) must be duality invariant. Since (a_D, a) transform under $SL(2, \mathbb{Z})$ as

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to M \begin{pmatrix} a_D \\ a \end{pmatrix} \ , \quad M \in SL(2, \mathbb{Z}) \ , \tag{73}$$

we must impose the following transformation on the electric and magnetic charges to render (72) duality invariant:

$$(n_m \, , n_e) \to (n_m \, , n_e) M^{-1} \ . \tag{74}$$

Now, we come back to the issue of constant shifts of the vector $\begin{pmatrix} a_D \\ a \end{pmatrix}$ which preserves the form of the Kähler metric. Such a shift changes the form of the central charge and hence changes the particle masses. Since duality transformation just changes the description of the low-energy theory to a different one, particle masses should not be affected. This rules out such constant shifts as there is no way to compensate for this by changing the charges (n_m, n_e).

7.2 Monodromies in the finite u-plane

Let us resume our discussion of monodromies in the case of two branch points in the finite u-plane at $u = \pm u_0$. The masses of BPS dyons are given by

$$M = |Z| = |an_e + a_D n_m| \ . \tag{75}$$

Hence, the monopole with charges $(n_m, n_e) = (1, 0)$ has zero mass when $a_D(u) = 0$.

We recall that the low-energy $U(1)$ theory which was weakly coupled at $u \to \infty$ becomes strongly coupled for smaller values of $|u|$. Also, the $SL(2,\mathbb{Z})$ transformation S exchanges a strongly coupled description with a weakly coupled description of the $U(1)$ theory while simultaneously exchanging the electric and magnetic charges. From the BPS formula, we inferred that magnetic monopoles become light in the vicinity of a zero of $a_D(u) = 0$. All these facts lead us to make the following ansatz for the low-energy theory at $u = u_0$:

Ansatz: The correct, weakly-coupled low-energy description in the vicinity of $u = u_0$ is an $\mathcal{N} = 2$ $U(1)$ gauge theory coupled to a charged, light BPS hypermultiplet which corresponds to the the magnetic monopole hypermultiplet in the original description.

We thus have an $\mathcal{N} = 2$ $U(1)$ gauge theory described by a gauge multiplet $\mathcal{A}_D$ coupled to a massive hypermultiplet Φ_D with unit electric charge under the gauge field $(A_D)_\mu$ in $\mathcal{A}_D$. One can calculate the 1-loop β-function for the running coupling $g_D(\mu)$ to obtain

$$\beta(g_D(\mu)) = \mu \frac{\mathrm{d}g_D}{\mathrm{d}\mu} = \frac{g_D^3}{8\pi} \ . \tag{76}$$

This is exact in perturbation theory due to $\mathcal{N} = 2$ supersymmetry. The positive sign of the β-function implies that the theory is weakly coupled in the infrared; this is consistent with our ansatz.

Manipulations similar to the $SU(2)$ SYM β-function give

$$\tau_D(a_D) = -\frac{i}{\pi} \log \frac{a_D}{\Lambda_D} \ , \tag{77}$$

where Λ_D is an RG invariant energy scale. Since a_D is assumed to be a good coordinate for the moduli space in the vicinity of $u = u_0$, one can write

$$a_D(u) = c_0(u - u_0) + \mathcal{O}((u - u_0)^2) \tag{78}$$

Since $\tau_D = \mathcal{F}''(a_D) = -1/\mathcal{F}''(a) = -\mathrm{d}a/\mathrm{d}a_D$, we get

$$\begin{aligned} a &= a_0 + \frac{i}{\pi}\left(a_D \log \frac{a_D}{\Lambda_D} - a_D\right) \\ &= a_0 + \frac{ic_0}{\pi}(u - u_0) \log \frac{c_0(u - u_0)}{\Lambda_D} + \mathcal{O}\Big((u - u_0)^2 \log(u - u_0)\Big) \ . \end{aligned} \tag{79}$$

Thus, under a 2π circuit around $u = u_0$, we have

$$a_D \to a_D \ , \quad a \to a - 2a_D \ . \tag{80}$$

This gives the monodromy matrix M_{u_0} and by using the relation $M_{u_0} M_{-u_0} = M_\infty$, also M_{-u_0}:

$$M_{u_0} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \ , \quad M_{-u_0} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \ , \quad M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \ . \tag{81}$$

The action on the charge vector (n_m, n_e) is given by

$$\begin{aligned} M_\infty \ &: \ n_m \to -n_m \ , \qquad && n_e \to 2n_m - n_e \ , \\ M_{u_0} \ &: \ n_m \to n_m + 2n_e \ , && n_e \to n_e \ , \\ M_{-u_0} &: \ n_m \to 3n_m + 2n_e \ , && n_e \to -2n_m - n_e \ . \end{aligned} \tag{82}$$

A monodromy around a branch point should not change the physical description of the theory in the vicinity of the branch point. We check this at each of the branch points.

1. The monodromy around infinity just gives an extra electric charge to magnetically charged particles, but these are all heavy in the regime $u \to \infty$. So the low-energy description in terms of $U(1)$ gauge theory without additional matter is not changed.

2. Similarly, the monodromy around u_0 acts as identity on the monopole $(n_m, n_e) = (1, 0)$, which is sensible since the low-energy theory consists of these light monopoles. Electrically charged particles get extra magnetic charge upon monodromy, but these are very heavy near $u \to u_0$ and hence do not change the low-energy description.

3. Continuing this reasoning, one can look for a dyonic object that is preserved by the monodromy around $u = -u_0$. It is not hard to see that a dyon with charges $(n_m, n_e) = (1, -1)$ satisfies this criterion. *Thus, a dyon with charges $(1, -1)$ should become massless at $u = -u_0$.*

Dyon democracy:
Since the magnetic monopole belongs to a BPS hypermultiplet, it saturates the mass bound $M \geq |Z|$. Hence, we expect it to be stable, at least semiclassically. The Witten effect tells us that by shifting θ by $2\pi n$ we get a whole series of dyons for free, with charges $(1, n)$, $n \in \mathbb{Z}$. Thus, the semiclassically stable solitonic objects in the BPS spectrum are the dyons with $(n_m, n_e) = (1, n)$. This indicates that there must be complete democracy among the $(1, n)$ dyons in the treatment above.

Let us look at the consistency condition on the monodromies:

$$M_{u_0} M_{-u_0} = M_\infty \ . \tag{83}$$

This condition is preserved under conjugation by $(M_\infty)^k$ for some integer k. The new matrices $M'_{\pm u_0} = (M_\infty)^k M_{\pm u_0} (M_\infty)^{-k}$ are given by:

$$M'_{u_0} = \begin{pmatrix} 1+4k & 8k^2 \\ -2 & 1-4k \end{pmatrix} , \quad M'_{-u_0} = \begin{pmatrix} -1+4k & 2-8k+8k^2 \\ -2 & 3-4k \end{pmatrix} . \tag{84}$$

A quick calculation shows that M'_{u_0} preserves the charges of a $(1, 2k)$ dyon while M'_{-u_0} preserves a $(1, 2k-1)$ dyon and hence it is these particles that become massless at $u = \pm u_0$. This is expected since the electric charge of a magnetic monopole changes precisely by $2k$ if one goes k times around $u = \infty$ due to the Witten effect. Thus, we indeed see that there is complete democracy among the dyons of the form $(1, n)$ for $n \in \mathbb{Z}$ and our assumption that monopoles become massless at $u = u_0$ is unique only upto such considerations.

7.3 Determining $a(u)$ and $a_D(u)$

The treatment in this section will closely follow the review by Bilal [Bi]. Using the minimal assumption of two branch points in the finite u-plane and physically plausible arguments, we have determined the monodromies of the functions $a_D(u)$ and $a(u)$ on the extended u-plane. By choosing a particular renormalization prescription, we can choose the branch points $u = \pm u_0$ to be at $u = \pm 1$.

The problem is then to conjure two functions $a_D(u)$ and $a(u)$ which possess the monodromies M_∞ and $M_{\pm 1}$ at $u = \infty$, $u = \pm 1$ respectively. Such functions usually arise as solutions of a second-order differential equation on the u-plane with regular singular points at $u = \infty, 1, -1$. A typical differential equation with at most second order poles at $u = \infty, 1, -1$ whose solution has monodromies is given by

$$\begin{gathered} -\frac{\mathrm{d}^2 f(u)}{\mathrm{d}u^2} + V(u) f(u) = 0 , \\ \text{with} \quad V(u) = -\frac{1}{4}\left[\frac{1-\lambda_{-1}^2}{(u+1)^2} + \frac{1-\lambda_1^2}{(u-1)^2} - \frac{1-\lambda_{-1}^2 - \lambda_1^2 + \lambda_\infty^2}{(u+1)(u-1)}\right] . \end{gathered} \tag{85}$$

The residues at $u = \infty, \pm 1$ are then given by $-\frac{1}{4}(1-\lambda_\infty^2)$ and $-\frac{1}{4}(1-\lambda_{\pm 1}^2)$ respectively. The two linearly-independent solutions of the differential equation will then possess monodromies around the regular singular points $u = \infty, \pm 1$. These monodromies depend on the exact behaviour of the solutions which depend on the precise values of the λ_i. Thus, if we fix the λ_i appropriately, one could recover the functions $a_D(u)$ and $a(u)$ with their monodromies.

It turns out this can indeed be done. We fix the λ_i by looking at the asymptotic behaviour of the solutions to (85). For $u \to \infty$, we see that the potential becomes $-\dfrac{1-\lambda_\infty^2}{4u^2}$. In this region, the two solutions then behave as $u^{(1\pm\lambda_\infty)/2}$ for $\lambda_\infty \neq 0$ and as $\sqrt{u}$ and $\sqrt{u}\log u$ for

$\lambda_\infty = 0$. We recognise the latter as the semiclassical formulas for $a(u)$ and $a_D(u)$ in (63) valid as $u \to \infty$. Thus, we fix $\lambda_\infty = 0$. Similarly, by comparing the formulas for a_D and a near $u = 1$, we can fix $\lambda_1 = 1$. Thus, the potential does not have second order poles at $u = \infty$ and at $u = 1$. By the $u \to -u$ symmetry of the problem, the potential cannot have a second order pole at $u = -1$ either. Thus, the final form of the potential relevant for our problem is

$$V(u) = \frac{-1}{4(u+1)(u-1)} \ . \tag{86}$$

The above differential equation can then be transformed into the hypergeometric differential equation

$$z(1-z)\,h''(z) + \big[c - (a+b+1)z\big]h'(z) - ab\,h(z) = 0 \ , \tag{87}$$

with solutions expressed in terms of the hypergeometric functions $F(a, b, c; z)$. A convenient choice is to pick one solution to have simple monodromy properties around $z = 1$ and the other solution to have simple monodromy around $z = \infty$ (in analogy with the behaviour of a_D and a):

$$\begin{aligned} h_1(z) &= (1-z)^{c-a-b}\,F(c-a, c-b, c+1-a-b; 1-z) \ , \\ h_2(z) &= (-z)^{-a}\,F(a, a+1-c, a+1-b; z^{-1}) \ . \end{aligned} \tag{88}$$

In our case, the constants a, b and c have values $a = b = -1/2$ and $c = 0$. Mapping these solutions back to the solutions of (85), the solution for a_D and a can be written as

$$\begin{aligned} a_D(u) &= \frac{i}{2}(u-1)\,F\left(\frac{1}{2}, \frac{1}{2}, 2; \frac{1-u}{2}\right) \ , \\ a(u) &= \sqrt{2(u+1)}\ F\left(-\frac{1}{2}, \frac{1}{2}, 1; \frac{2}{u+1}\right) \ . \end{aligned} \tag{89}$$

Using the properties of hypergeometric functions, it can be checked that the above solutions indeed have the monodromies M_∞, $M_{\pm 1}$ around $u = \infty, \pm 1$ respectively. Using the integral representation of hypergeometric functions, one can rewrite a_D and a in the following simpler form:

$$a_D(u) = \frac{\sqrt{2}}{\pi}\int_1^u \mathrm{d}x\,\frac{\sqrt{x-u}}{x^2-1} \ , \quad a(u) = \frac{\sqrt{2}}{\pi}\int_{-1}^1 \mathrm{d}x\,\frac{\sqrt{x-u}}{x^2-1} \ . \tag{90}$$

This representation will be useful in the next section, where we give a geometric interpretation to the above solution for $(a_D(u), a(u))$ and the formula for the central charge $Z = -an_e - a_D n_m$.

To summarise, using the monodromy properties of $a_D(u)$ and $a(u)$ that we inferred from physical considerations, we were able to obtain exact forms of $a_D(u)$ and $a(u)$. One could now invert the formula for a to get u in terms of a and plug it into $a_D(u)$ to get a_D in terms of a. One can then integrate a_D with respect to a to obtain the prepotential in the weakly coupled regime, $\mathcal{F}(a)$. In particular, one can obtain all the coefficients $\mathcal{F}_k$ in the instanton sum.

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A A technical note on the $\mathcal{N} = 2$ β-function

In the treatment by Seiberg in "Supersymmetry and Non-perturbative beta functions", Phys. Lett. **B206** (1988) 75, *it is assumed, and later proven, that* the homogeneous $U(1)_{\rm R}$ transformation $\mathcal{A} \to e^{2i\alpha}\mathcal{A}$ continues to be homogeneous in the quantum theory as well. Then, the invariance of the effective action under $U(1)_{\rm R}$ R-symmetry is used to show that the most general form of the perturbative prepotential is given by

$$\mathcal{F}(\mathcal{A}) = \mathcal{A}^2 \left[a_1 + a_2 \log \frac{\mathcal{A}^2}{\Lambda^2} \right] , \tag{91}$$

where a_1 can be eliminated by a rescaling of Λ and a_2 has to be determined separately. The same argument given at the beginning of Section 4.1 on R-symmetry shows that the effective Lagrangian transforms inhomogeneously under $U(1)_{\rm R}$ due to the logarithmic term. This can be recognised as the perturbative anomaly which tells us that the log term is, in fact, a one-loop effect.

Next, using this form of the effective action, one can extract the β-function by using $\tau_{\rm eff} = \mathcal{F}''$ and differentiating with respect to the symmetry breaking scale a. Since the action was determined to all orders in perturbation theory, the β-function obtained from such an analysis is also perturbatively exact and is given by the one-loop contribution.

Now, an important point is made by Seiberg. In $\mathcal{N} = 1$ SYM theory, unlike the above case, the $\mathcal{N} = 1$ chiral superfield (the analog of A) which appears in the quantum action alongside the running coupling usually *does not continue* to have nice (homogeneous) transformation properties under $U(1)_{\rm R}$. In particular, the form of the effective action is no longer as severely restricted as above, and the β-function may receive higher-loop contributions while the axial anomaly is still only a 1-loop effect. This is a manifestation of the anomaly puzzle in $\mathcal{N} = 1$ SYM theory.

One could always rewrite the action in terms of a new chiral superfield $\widetilde{\mathcal{A}}$ which has homogeneous $U(1)_{\rm R}$ transformation *through a complicated, supersymmetric field redefinition* such that the β-function is now one-loop exact in these new variables. However, such field redefinitions are usually singular at $\widetilde{\mathcal{A}} = 0$, and hence this does *not* imply that the original β-function is one-loop exact as well.

In $\mathcal{N} = 2$ SYM theory, such a redefinition is not possible while being consistent with $\mathcal{N} = 2$ supersymmetry. Hence, there is no candidate other than $\mathcal{A}$ for a quantum $\mathcal{N} = 2$ chiral superfield that appears alongside the running coupling and also continues to have a homogeneous $U(1)_{\rm R}$ transformation. This ensures that the β-function is one-loop exact iff the axial anomaly is one-loop exact.